$\min f(x)$ s.t. $C_i(x) > 0 \quad i \in \mathcal{E}$

$C_i(x) = 0 \quad i \in \mathcal{L}$

KKT Condition

x^* : local solution

$\exists \lambda^*$

(1) $\nabla L_x(x^*, \lambda^*) = 0$

(2) $C_i(x^*) = 0 \quad i \in \mathcal{E}$

(3) $C_i(x^*) > 0 \quad i \in \mathcal{L}$

(4) $\lambda_i^* \geq 0$

(5) $\lambda_i^* C_i^{(x^*)} = 0$

~~Thm:~~

~~$T_\Omega(x^*) = \mathcal{F}(x^*)$~~

Lemma: feasible x^*,

$T_\Omega(x^*) \subset \mathcal{F}(x^*)$ ①

$LIQC^{Q}(x^*) = 1 \Rightarrow T_\Omega(x^*) = \mathcal{F}(x^*)$ ②

Proof: Taylor Thm $\Rightarrow$ Implicit Function Thm.

$C_i(x)$, $i = 1..m$ are active at x^*

(i) for each direction $d \in T_\Omega(x^*)$

$\exists$ limiting sequence $\{z_k\}$, $\{t_k\}$

$d = \lim_{k \to \infty} \frac{z_k - x^*}{t_k}$

$i \in \mathcal{E}$: $0 = \frac{1}{t_k} C_i(z_k)$

$= \frac{1}{t_k} C_i(x^*) + \nabla C_i^T(x^*)(z_k - x^*) + O(\|z_k - x^*\|^2)$

$= \nabla C_i^T(x^*) d + O(1) \Rightarrow$

$\nabla C_i^T(x^*) d = 0$

$i \in A(x^*) \cap \mathcal{L}$: $\nabla C_i^T(x^*) d > 0$

$\therefore d \in \mathcal{F}(x^*)$

(ii) $A(x^*) = [\nabla C_i(x^*), i \in A(x^*)]$ $m \times n$

LICQ $\Rightarrow A(x^*)$ is full rank

Construct:

$$R(z,t) = \begin{bmatrix} C(z) - tA(x^*)d \\ z^T(z - x^* - td) \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\nabla R_z(x^*, 0) = \begin{bmatrix} A(x^*) \\ z^T \end{bmatrix}$$ is nonsingular

$f(x,t) \approx f(x^*, 0) + f_x(x^*, 0)(x - x^*) + \frac{\partial f}{\partial t}(x^*, 0) t$

must $\exists z, t$, s.t. $R(z,t) = 0$:

$$R(z,t) = \begin{bmatrix} A(x^*) \\ z^T \end{bmatrix} (z_k - x^* - t_k d) + O(\|z_k - x^*\|)$$

$$= \begin{bmatrix} A(x^*)(z_k - x^*) + O(\| \ \|) - t_k A(x^*) d \\ z^T(z_k - t^* - td) \end{bmatrix} = 0$$

$$\frac{z_k - x^*}{t_k} = d + O\left(\frac{\| \ \|}{t_k}\right)$$

Foundamental Necessary Condition

Thm. x^* is a local solution.

$\nabla f(x^*)^T d \geq 0$ for $d \in T_\Omega(x^*) \overset{\text{LICQ}}{\Rightarrow} \mathcal{F}(x^*)$

Farkas Thm:

$K = \{\overset{n \times m}{B} y + \overset{n \times p}{C} w \mid y \geq 0\}$

(1) $g \in K$

(2) $\exists d$, $g^T d < 0$, $B^T d \geq 0$ $C^T d = 0$

Proof: these 2 statements cannot hold simutaneously:

$$0 > g^T d = (By + Cw)d \geq 0$$

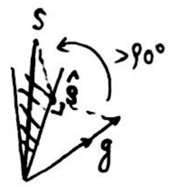
$$\hat{s} = \underset{s \in K}{\arg\min} \, \| s - g \|^2$$

$$d = \hat{s} - g$$

$$0 = \frac{d}{d\alpha} \| \alpha \hat{s} - g \| \Big|_{\alpha = 1}$$

$$\Rightarrow \hat{s}(\hat{s} - g) = 0 \quad \cdots\cdots \quad (1)$$

$$(s - \hat{s})(\hat{s} - g) > 0$$



$$d^T g = (\hat{s} - g)^T g = (\hat{s} - g)^T \hat{s} - \| \hat{s} - g \|^2 < 0$$

$$d^T (By + Cw) \geq 0 \Rightarrow \cancel{d^T(By) \geq 0}$$

$$B^T d \geq 0$$

$$C^T d = 0$$

Apply Farkas Thm to Cone:

$$N = \left\{ \sum_{i \in A(x^*)} \lambda_i \nabla c_i(x^*), \ \lambda_i \geq 0 \text{ for } A \cap \mathcal{I} \right\}$$

set $g = \nabla f(x^*)$

(1) $\nabla f(x^*) = \sum_{i \in A} \lambda_i \nabla c_i(x^*) = A(x^*)^T x^*$

(2) $\exists d, \ d^T \nabla f(x^*) < 0, \ d \in \mathcal{F}(x)$

not true under LICQ